Stochastic quantization of Yang–Mills

(and a gentle survey of Yang–Mills)

Hao Shen (Wisconsin-Madison)

Coauthors: Ajay Chandra, Ilya Chevyrev, Martin Hairer
Scott Smith, Rongchan Zhu, Xiangchan Zhu

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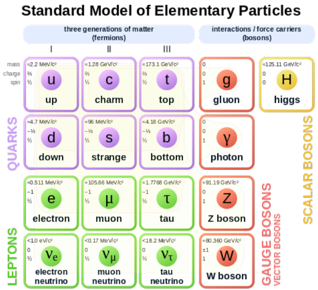
- Yang, Mills (1953):
  $U(1)$ gauge symmetry in Maxwell $\to$ general symmetries
- To allow general gauge symmetries,
  Vector fields in Maxwell theory $\to$ **Lie algebra** valued 1-forms;
  Linear equations in Maxwell theory $\to$ **nonlinear** ones.
- YM is now the unified framework for elementary particles.
  EM, weak, strong forces $= U(1) \times SU(2) \times SU(3)$ YM in 4D.

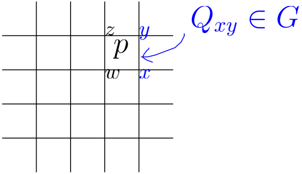


- Recall: **classical** model defined by an action functional $\mathcal{S}(X)$, the **quantum** theory is formulated as functional integrals:

$$\int (\cdots) e^{-\mathcal{S}(X)} dX$$

- To define the YM model,
  let's fix a compact Lie group G (e.g. $G = U(N)$),
  and $\mathfrak{g}$ is its Lie algebra (e.g. $\mathfrak{g} = \{N \times N \text{ skew-Hermitians}\}$).
  Recall the exponential map $Exp : \mathfrak{g} \to G$.

Lattice Yang–Mills model:



$$\mathcal{S}(Q) = \sum_p Tr(Q_{xy}Q_{yz}Q_{zw}Q_{wx})$$

Measure (well-defined!) $e^{-\mathcal{S}(Q)}dQ_{Haar}$

$\forall g : lattice \to G$, $\mathcal{S}(Q)$ *is invariant under* $Q_{xy} \mapsto g_x Q_{xy} g_y^{-1}$.

Yang–Mills in continuum: $A = (A_1, \cdots, A_d)$, with $A_i : \mathbf{R}^d \to \mathfrak{g}$.

$$\mathcal{S}(A) = \int \|F_A\|^2 dx \qquad F_A^{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$$

"Measure" (completely formal!): $e^{-\mathcal{S}(A)}DA$

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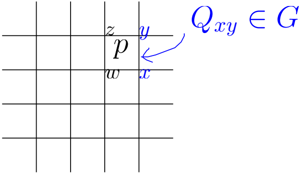

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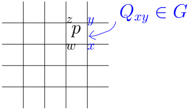
Remarks:

- If $d = 4$, $G = U(1)$, Euler-Lagrange equ $\to$ **Maxwell equ**:
  $F^{ij} = \partial_i A_j - \partial_j A_i$ has 6 components,
  3 correspond to electric field, 3 correspond to magnetic field.
- **Geometry:** The field A corresponds to *connections on bundles* in geometry; a connection is used to specify "a way to differentiate bundle sections".



Covariant derivative of Φ:
$(d_Q \Phi)_x \stackrel{\text{def}}{=} Q_{xy}\Phi_y - \Phi_x$

In continuum, covariant derivative $d_A \Phi \stackrel{\text{def}}{=} d\Phi + A\Phi$.

Curvature F_A measures "non-flat-ness",
and $\mathcal{S}(A) = \int \|F_A\|^2 dx$ is a natural functional.

- Yang–Mills–Higgs model $\mathcal{S}(A, \Phi) \stackrel{\text{def}}{=} \int \|F_A\|^2 + |d_A \Phi|^2 dx$

Important observables in Yang–Mills theory

Lattice:

For a lattice loop $\gamma = x_1 x_2 \cdots x_{n-1} x_n x_1$,

$$Tr(Q_{x_1x_2} Q_{x_2x_3} \cdots Q_{x_{n-1}x_n} Q_{x_nx_1})$$

is called a Wilson loop. It is a gauge invariant observable.

Continuum:

Given $\gamma : [0,1] \to \mathbf{R}^d$, $\gamma(0) = \gamma(1)$, Wilson loop is defined by

$$Tr(h(1))$$

which is gauge invariant, where h is "holonomy" along γ:

$$dh(s) = h(s)\langle A(\gamma(s)), d\gamma(s)\rangle \qquad h(0) = \mathrm{id} \in G$$

Special (and very interesting) case: $d = 2$

While $e^{-\mathcal{S}(A)} DA$ is generally hard to make sense in continuum, in $d = 2$ it is a **well-defined** theory and **integrable** theory.

What's special at $d = 2$ is:
By a gauge transformation $(A_1, A_2) \to (0, \tilde{A}_2)$, and $F^{12}_{\tilde{A}} = \partial_1 \tilde{A}_2$.
Then $S(A)$ is quadratic in A and $e^{-\mathcal{S}(A)} DA$ becomes Gaussian.
(We're allowed to compute correlation of gauge invariant observables in a 'nice' gauge)

It is integrable (correlations have exact formulas), for example:

$$\mathbf{E}[\text{Wilson loop of a simple loop } \gamma] = \exp(\text{Area}(\gamma)/2)$$

[Driver, Gross, King, Lévy, Sengupta... '90s] [Park–Pfeffer–Sheffield–Yu'23]

Also, Wilson loop correlations satisfy interesting recursions.
[Levy'11], [Dahlqvist'15], [Driver–Hall–Kemp'17]

To recap,

- In 2D continuum ($\mathbf{R}^2$ or surfaces), YM measure is well-defined. There have been many studies on correlation formulas, large N behavior ($G = U(N)$), etc.
- On finite lattice, any dimension, Yang–Mills measure is well-defined. There are also studies on topics such as:
  - "loop equations" (i.e. recursions of Wilson loop correlations),
  - large N behavior,
  - decay of two-point correlations, decay of large Wilson loops,
  - duality with (discrete) random surface models, etc.

  (e.g. [Osterwalder–Seiler'78] [Borgs–Seiler'83] [Chatterjee'17] [Cao–Park–Sheffield'23]....)
- In continuum, 3D and 4D, YM measure is not yet rigorously defined.

Difficulties to make sense and study " $e^{-\mathcal{S}(A)} DA$ "

1. **Small scale singularity** (ultraviolet problem, for all QFTs): Even the Gaussian free field $e^{-\int |\nabla\Phi|^2 dx} D\Phi$ is supported on $\mathcal{C}^\alpha$ for $\alpha < -(d-2)/2$.
   So when $d \geq 2$, the fields are not functions (they are "generalized functions" a.k.a. "distributions"), and so the products e.g. $[A_i, A_j]$ do not have classical meaning.
   (This is related with UV divergence and renormalization.)

2. **Meaning of gauge invariant observables**
   Given $\gamma : [0,1] \to \mathbf{R}^d$, $\gamma(0) = \gamma(1)$,
   Wilson loop observables $W_\gamma(A) \stackrel{def}{=} Tr(h(1)) \in G$ are gauge invariant, where h is "holonomy" along γ:

$$dh(s) = h(s)\langle A(\gamma(s)), d\gamma(s)\rangle \qquad h(0) = \mathrm{id} \in G$$

   But generic distribution in $\mathcal{C}^\alpha$ for $\alpha < 0$ can not be integrated along curves.

3. **Gauge invariance, or Gauge fixing:**
   The formal measure $e^{-\mathcal{S}(A)}DA$ should be defined on $\{A\}/\sim$, where $\sim$ is gauge equivalence. Recall that in smooth setting gauge equivalence is defined by $A \sim gAg^{-1} - (dg)g^{-1}$.

   What do these mean in the singular setting?

   "Gauge fixing": finding a "cross-section", i.e. selecting a representative from each gauge equivalence class.
   In abelian case, one could fix e.g. div $A = 0$.
   In non-abelian case: "Gribov ambiguity" [Singer'78]
   (other issues like "ghost" anti-commuting variables)

4. **Large field behavior:**
   There are simpler non-Gaussian QFT

$$e^{-\int(|\nabla\Phi|^2+\Phi^4)dx}D\Phi\ , \qquad e^{-\int(|\nabla\Phi|^2+e^{\gamma\Phi})dx}D\Phi$$

   where "stability" mechanism is clearer, i.e. probability becomes small when Φ is large. This is very unclear for YM.

5. **Osterwalder–Schrader axioms ('70s):**
   If correlation functions satisfy some axiomatic properties, then one can reconstruct the quantum mechanics (Hilbert space and operators) on Minkowski space.
6. **Long distance ("infrared") questions**
   For instance:
   "Mass gap": correlations decay exponentially?
   "Quark confinement": Wilson loop expectation decays like area versus perimeter law?
   Such long distance problems are often studied with lattice YM models, to isolate difficulties from small scales. (I will comment in the end.)

1. **Small scale singularity (ultraviolet problem)**
2. **Mathematical meaning of gauge invariant observables**
3. **Mathematical meaning of gauge invariance / equivalence**
4. **Control of large field behavior**
5. **Verify properties such as Osterwalder–Schrader axioms**
6. **Long distance problems (e.g. exponential decay)**

[Chandra–Chevyrev–Hairer–S. '20, '22], [Chevyrev–S. '23]

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On 2D, 3D tori, we focus on questions 1,2,3. In particular:

- **Deterministic construction:**
  We constructed new spaces of singular 1-forms (can be embedded into standard Sobolev/Besov spaces), on which one has notions of gauge equivalence and gauge invariant observables.
- **Stochastic construction:**
  In the "stochastic quantization" setting, small scale singularities can be well-understood, and we obtain a Markov process on "quotient space" $\{A\}/\sim$.

Stochastic quantization

- In finite dimensions, it's well-known in probability theory that

$$dX_t = -\nabla V(X_t)dt + dB_t \qquad X_0 \in \mathbf{R}^n$$

where B is Brownian motion in $\mathbf{R}^n$,
the Prob Distribution of X_t goes to $e^{-V(x)}dx$ as $t \to \infty$.

- Infinite dimensional analogue: the $t \to \infty$ distribution of

$$\partial_t A = -\nabla \mathcal{S}(A) + \xi$$

"would give a meaning" to $e^{-\mathcal{S}(A)} DA$.
Here ξ is Gaussian white noise, with $\mathbf{E}[\xi(t,x)] = 0$ and

$$\mathbf{E}[\xi_i(t,x)\xi_i(t',x')] = \delta(t-t')\delta(x-x').$$

Remark: ξ is very singular! ($\xi \in \mathcal{C}^{-(d+2)/2-}$ almost surely)

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Stochastic quantization

The simplest (yet nontrivial) model is Φ^4 model:

$$e^{-\int(|\nabla\Phi|^2+\Phi^4)dx} D\Phi$$

Its stochastic quantization is the following SPDE:

$$\partial_t\Phi = \Delta\Phi - \Phi^3 + \xi$$

Stochastic quantization of Φ^4 in $d = 3$ has been successfully done:

[Hairer'13]: Local solution. $(\partial_t\Phi_\varepsilon = \Delta\Phi_\varepsilon - \Phi_\varepsilon^3 + C_\varepsilon\Phi + \xi_\varepsilon,\ \varepsilon \to 0)$
Solved "small scale problem"
[Mourrat–Weber'17]: Obtained time-independent bound as $t \to \infty$.
Krylov–Bogoliubov argument to construct the invariant measure.
[Gubinelli–Hofmanova'17]: Consider stationary solution on lattice. The fixed-time law is tight as lattice spacing vanishes.
Solved "large field problem"

Stochastic quantization of Yang–Mills

The dynamic $\partial_t A = -\nabla \mathcal{S}(A) + \xi$ has the following form

$$\partial_t A_i = \Delta A_i + [A_j, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]] + \xi_i \quad (i = 1, ..., d)$$

Remark. **[Parisi–Wu '81 "Perturbation theory without gauge fixing"]:**

- The linear part of the dynamic is degenerate and not ΔA_i, but one can add a drift tangent to the gauge transformation to get ΔA_i.
- This is "local gauge tuning" instead of "global gauge fixing".
- This is called DeTurck trick in (deterministic) geometric flows.

The stochastic PDE is also formal due to small scale singularity.
We obtained local solution using Hairer's theory in $d = 2, 3$.
(For large t, large field problem would show up.)

Hairer's theory of regularity structures (2013):
Given a subcritical (or "super-renormalizable") equation,

$$\{\text{a finite collection of "perturbative objects"}\} \to \text{solution}$$

is a **continuous** map.

Renormalized stochastic Yang–Mills equation:

$$\partial_t A_i = \Delta A_i + [A_j, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]] - CA_i + \xi_i$$

If $d = 2$: A_0^3, $P(A_0 \partial A_0)(\partial A_0)$, $\partial P(A_0 \partial A_0) A_0$ (where $P = (\partial_t - \Delta)^{-1}$)
If $d = 3$: hundreds of "perturbative objects".
If $d = 4$: infinitely many.

Gauge equivalence

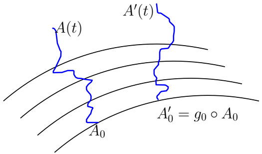
$d = 3$: we constructed a (nonlinear) space $\mathcal{S} \subset \mathcal{C}^{-\frac{1}{2}-}$, such that

- $\forall A \in \mathcal{S}$, we consider a *gauge invariant* regularization A_δ ($\delta > 0$) obtained by solving deterministic Yang-Mills heat flow with initial condition A up to "time" $\delta > 0$.

  (Similar result [Cao–Chatterjee'21]; idea from [Charalambous–Gross'13])
- the Wilson loop $\mathrm{Tr}\,\mathrm{hol}_\ell(A_\delta)$ is a gauge invariant observable.
- Define gauge equivalence $A \sim \bar{A}$ on $\mathcal{S}$ if $A_\delta \sim \bar{A}_\delta$ for $\delta > 0$.

Project solution to the quotient space $\mathcal{S}/\sim$

$$\partial_t A_i = \Delta A_i + [A_j, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]] - CA_i + \xi_i$$



There is a finite shift of C such that the limit is gauge covariant, which allows us to project the solution to $\mathcal{S}/\sim$

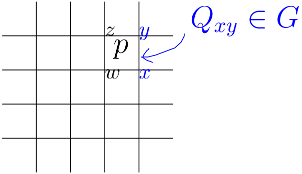
In 2D continuum, or on lattice, we can prove many properties of YM measure using the dynamic.

In 2D continuum:

[Chevyrev–S.'23] 2D YM measure is indeed the unique invariant measure for the YM stochastic dynamic. ("Bourgain argument".)

This characterization has many applications: e.g. 2D YM measure is the universal limit of different lattice approximations.

On lattice of dimension d:



$$e^{-\beta \mathcal{S}(Q)} dQ_{Haar}$$

where $\mathcal{S}(Q) = \sum_p Tr(Q_p)$

and $Q_p \stackrel{\text{def}}{=} Q_{xy} Q_{yz} Q_{zw} Q_{wx}$.

[S.,Smith,Zhu,Zhu'22-'24]

Lattice dynamic ($G = U(N)$ case): at each edge e,

$$dQ_e = \beta \sum_{p \supset e} (Q_p - Q_p^*) dt + d\mathfrak{B}_e$$

where $(\mathfrak{B}_e)$ are i.i.d. Brownian motions on $U(N)$.

- Re-derive master loop equation by Ito formula.
- One can prove exponential decay of correlation ("mass gap") for small β using the dynamic and log-Sobolev.
  (Monday talks: dynamic on manifolds, mixing, coupling, curvature....)
- Large N limit. etc...

Open questions:

- "Large field problem": In 3D, how to obtain strong enough bound on $A(t)$ as t becomes large?
- "Long distance problem":
  Construct the dynamic on $\mathbf{R}^3$ instead of $\mathbf{T}^3$.
  Prove mass gap in *continuum* in 3D. (Or large β on lattice.)
- Anything can be said in 4D?